

An Elementary Proof of the Polynomial Matrix Spectral Factorization Theorem

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Abstract. A very simple and short proof of the polynomial matrix spectral factorization theorem (on the unit circle as well as on the real line) is presented, which relies on elementary complex analysis and linear algebra.

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1. INTRODUCTION

In this paper, we present an elementary proof of the polynomial matrix spectral factorization theorem:

Theorem 1. *Let*

$$S(z) = \sum_{n=-N}^N C_n z^n$$

be an $m \times m$ matrix function ($C_n \in \mathbb{C}^{m \times m}$ are matrix coefficients) which is positive definite almost everywhere on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Then it admits a factorization

$$S(z) = S^+(z)S^-(z), \quad z \in \mathbb{C} \setminus \{0\},$$

where $S^+(z) = \sum_{n=0}^N A_n z^n$ is an $m \times m$ polynomial matrix function which is nonsingular inside \mathbb{T} , $\det S^+(z) \neq 0$ when $|z| < 1$, and $S^-(z) = \overline{S^+(1/\bar{z})}^T = \sum_{n=0}^N A_n^ z^{-n}$ is its adjoint, $A_n^* = \overline{A_n}^T$, $n = 0, 1, \dots, N$, (respectively, S^- is analytic and nonsingular outside \mathbb{T}). S^+ is unique up to a constant right unitary multiplier.*

In the scalar case, $m = 1$, the above result is known as the Fejér-Riesz lemma and can be easily proved by considering the zeroes of $S(z)$.

The matrix spectral factorization $S(t) = S^+(t)(S^+(t))^*$, $|t| = 1$, was first established by Wiener [8] in a general case for any integrable matrix-valued function $S(t)$, $|t| = 1$, with an integrable logarithm of the determinant, $\log \det S(t) \in L_1(\mathbb{T})$. In this case, the spectral factor S^+ belongs to the Hardy space H_2 . Wiener proved this theorem by using a linear prediction theory of multi-dimensional stochastic processes. A little bit later, by using the same methods, Rosenblatt [7] showed that S^+ is a polynomial whenever S is a Laurent polynomial. Since then, many different simplified proofs of the matrix spectral factorization theorem have appeared in the literature, see for example [9], [2], [1], [4]. A very short proof of this result is given in [3], however it uses Wiener's general spectral factorization existence theorem and some facts from the theory of the Hardy spaces.

The presented proof relies only on elementary complex analysis and linear algebra. This proof is constructive, which makes it possible to compute the spectral factor approximately at least in the case of low dimensional matrices (see [5] for a new reliable computational algorithm of general matrix spectral factorization). The same pattern can be also used for proving in a straightforward manner the polynomial matrix spectral factorization theorem on the real line:

Theorem 2. *Let*

$$S(z) = \sum_{n=0}^{2N} C_n z^n$$

be an $m \times m$ matrix function ($C_n \in \mathbb{C}^{m \times m}$ are matrix coefficients) which is positive definite almost everywhere on the real line \mathbb{R} . Then it admits a factorization

$$S(z) = S^+(z)S^-(z), \quad z \in \mathbb{C},$$

where $S^+(z) = \sum_{n=0}^N A_n z^n$ is an $m \times m$ polynomial matrix function which is non-singular in the open upper half plane, $\det S^+(z) \neq 0$ when $\text{Im} z > 0$, and $S^-(z) = \overline{S^+(\bar{z})}^T = \sum_{n=0}^N A_n^ z^n$ is its adjoint, $A_n^* = \overline{A_n}^T$, $n = 0, 1, \dots, N$, (respectively, S^- is nonsingular in the open lower half plane). S^+ is unique up to a constant right unitary multiplier.*

Various practical applications of spectral factorization in linear systems are widely recognized (see, e.g. [6]), where the problem naturally arises in either of the two different forms commonly called discrete and continuous. Mathematically both forms are equivalent under a conformal mapping of the upper half plane into the unit disk.

2. NOTATION

For $a \in \mathbb{C}$, $a^* = \bar{a}$ denotes its conjugate, and for a matrix A , $A^* = \overline{A}^T$ denotes its Hermitian conjugate.

Let $\mathbb{T}_+ = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T}_- = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, $\mathbb{R}_+ = \{z \in \mathbb{C} : \text{Im} z > 0\}$, and $\mathbb{R}_- = \{z \in \mathbb{C} : \text{Im} z < 0\}$.

$L_1^+(\mathbb{T})$ ($L_1^-(\mathbb{T})$) stands for the class of integrable functions on \mathbb{T} whose Fourier coefficients with negative (positive) indices equal to zero.

Let $\mathcal{R}^{m \times m}$ be the ring of rational $m \times m$ matrix functions defined in the complex plane. For $f \in \mathcal{R}^{m \times m}$, the *adjoint* matrix function \tilde{f} is defined by $\tilde{f}(z) = \overline{f(1/\bar{z})}^T$ in the discrete case and by $\tilde{f}(z) = \overline{f(\bar{z})}^T$ in the continuous case. Since f is uniquely determined by its values on \mathbb{T} (on \mathbb{R}), and $\tilde{f}(z) = (f(z))^*$ for $z \in \mathbb{T}$ (for $z \in \mathbb{R}$), usual relations for adjoint matrix functions, like $\widetilde{fg}(z) = \tilde{g}(z)\tilde{f}(z)$ and $\widetilde{f^{-1}}(z) = \tilde{f}^{-1}(z)$, etc., are valid. Obviously, if $f(e^{i\theta}) \in L_1^+(\mathbb{T})$, then $\tilde{f}(e^{i\theta}) \in L_1^-(\mathbb{T})$. Whenever $S^+ \in \mathcal{R}^{m \times m}$ is determined, S^- always denotes its adjoint.

$U \in \mathcal{R}^{m \times m}$ is called paraunitary if

$$U(z)\tilde{U}(z) = I_m$$

where I_m stands for the m -dimensional unit matrix. Note that $U(z)$ is a usual unitary matrix on the boundary, i.e.

$$(1) \quad U(z)U(z)^* = I_m, \quad z \in \mathbb{T} \quad (z \in \mathbb{R}).$$

We say that a matrix function is analytic in a domain if the entries of the matrix are analytic in the domain.

3. AN ELEMENTARY PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. By Gauss elimination on the matrix $S(z)$ and the Fejér-Riesz lemma, a factorization

$$(2) \quad S(z) = S_0(z) \tilde{S}_0(z)$$

can be easily achieved with $S_0(z) \in \mathcal{R}^{m \times m}$. (Namely, if $A = (a_{ij})_{i,j=1,\overline{m}}$ is a positive definite matrix and $B = (b_{ij})_{i,j=1,\overline{m}}$ is the unique positive definite matrix such that $A = BB^*$, then the entries of B can be recursively determined by the formulas $b_{11} = \sqrt{a_{11}}$, $b_{k1} = a_{k1}/\sqrt{a_{11}}^*$, $k = 2, 3, \dots, m$, $b_{nn} = \sqrt{a_{nn} - \sum_{j=1}^{n-1} b_{nj}b_{nj}^*}$, $b_{kn} = (a_{kn} - \sum_{j=1}^{n-1} b_{kj}b_{nj}^*)/b_{nn}^*$, $n = 2, 3, \dots, m$, $k = n+1, \dots, m$. These formulas remain valid for rational matrix functions as well. We need only to assume that \sqrt{a} and a^* are the scalar spectral factor of a and the adjoint of a , respectively.)

If s_{ij} is the ij th entry of S_0 with a pole at $a \in \mathbb{T}_+$, then we can multiply S_0 by the paraunitary matrix function $U(z) = \text{diag}[1, \dots, u(z), \dots, 1]$, where $u(z) = (z-a)/(1-\bar{a}z)$ is the jj th entry of $U(z)$, so that the ij th entry of the product $S_0(z)U(z)$ will not have a pole at a any longer keeping the factorization (2): $(S_0U)(z) \tilde{S}_0U(z) = S_0(z) \tilde{S}_0(z) = S(z)$. In the same way, one can remove every pole of the entries of S_0 at points in \mathbb{T}_+ . Thus S can be represented as a product

$$(3) \quad S(z) = S_0^+(z) S_0^-(z),$$

where $S_0^+ \in \mathcal{R}^{m \times m}$ is analytic in \mathbb{T}_+

Now, it might happen so that S_0^+ is not nonsingular everywhere on \mathbb{T}_+ . If $|a| < 1$ and $\det S_0^+(a) = 0$, then there exists an $m \times m$ unitary matrix U such that the product $S_0^+(a)U$ has all 0's in the first column. Hence a is a zero of every entry of the first column of the matrix function $S_0^+(z)U$ and the product $S_1^+(z) = S_0^+(z)U \text{diag}[u(z), 1, \dots, 1]$, where $u(z) = (1 - \bar{a}z)/(z - a)$, remains analytic inside \mathbb{T} . While the factorization (3) remains true replacing S_0^+ and S_0^- by S_1^+ and S_1^- , respectively, the determinant of S_1^+ will have less zeros in \mathbb{T}_+ than the determinant of S_0^+ . Thus, continuing this process if necessary, we can remove any singularities in \mathbb{T}_+ and get the factorization

$$(4) \quad S(z) = S^+(z) S^-(z),$$

where $S^+ \in \mathcal{R}^{m \times m}$ is analytic and nonsingular in \mathbb{T}_+ .

Now let us show that S^+ is in fact a polynomial matrix function of order N . S^+ is free of poles on \mathbb{T} since $S^+(z)(S^+(z))^* = S(z)$ for $z \in \mathbb{T}$, and $z^{-N}S^+(z) = z^{-N}S(z)(S^-(z))^{-1}$ is analytic in \mathbb{T}_- . Consequently S^+ is analytic in \mathbb{C} and $z^{-N}S^+(z)$ is analytic in \mathbb{T}_- which implies that S^+ is a polynomial of order N .

The proof of the uniqueness of S^+ is standard and it is given only for the sake of completeness. Namely if $S(z) = S_1^+(z)S_1^-(z) = S_2^+(z)S_2^-(z)$ are two spectral factorizations of $S(z)$, then $(S_2^+(z))^{-1}S_1^+(z)$ is an analytic and nonsingular in \mathbb{T}_+ paraunitary matrix function, which is free of poles and singularities on \mathbb{T} because of (1). Therefore $(S_2^+(e^{i\theta}))^{-1}S_1^+(e^{i\theta}) \in L_1^+(\mathbb{T}) \cap L_1^-(\mathbb{T})$, which implies that it is a constant matrix function.

Proof of Theorem 2. This proof can be carried out directly by the same steps as in the discrete case. The factorization (2) can be performed exactly in the same way as in the proof of Theorem 1 assuming under \sqrt{a} a scalar spectral factor of rational function a which is positive almost everywhere on \mathbb{R} . Elimination of poles of S_0 (see (2)) and singularities of S_0^+ (see (3)) in \mathbb{R}_+ can be made by using the paraunitary factors $u(z) = \frac{z-a}{z-\bar{a}}$ and $u(z) = \frac{z-\bar{a}}{z-a}$, respectively. When we have the factorization (4) where $S^+ \in \mathcal{R}^{m \times m}$ is analytic and nonsingular in \mathbb{R}_+ , we can prove that S^+ is in fact a polynomial as follows: S^+ is free of poles on \mathbb{R} since $S^+(z)(S^+(z))^* = S(z)$ for $z \in \mathbb{R}$ (as in the discrete case), and $S^+ = S(S^-)^{-1}$ is analytic in \mathbb{R}_- . Consequently $S^+ \in \mathcal{R}^{m \times m}$ is analytic in \mathbb{C} and hence polynomial. The order of S^+ is N since if A is a nonzero matrix coefficient of the highest order of z in S^+ , then $AA^* \neq 0$ is the matrix coefficient of the highest order of z in S^+S^- .

The problem of uniqueness of S^+ can be reduced to the discrete case by using the linear fractional transformation $z \rightarrow \frac{i+iz}{1-z}$ which maps \mathbb{T}_+ to \mathbb{R}_+ .

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